Self-Affine Sets with Positive Lebesgue Measure

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Abstract

Using techniques introduced by C. Güntürk, we prove that the attractors of a family of overlapping self-affine iterated function systems contain a neighbourhood of zero for all parameters in a certain range. This corresponds to giving conditions under which a single sequence may serve as a 'simultaneous β-expansion' of different numbers in different bases.

1 Introduction

Given real numbers $1 < \beta_1 < \beta_2$, we define contractions $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_i(x, y) = \left(\frac{x+i}{\beta_1}, \frac{y+i}{\beta_2} \right).$$

A classical result of Hutchinson [4] asserts that there exists a unique non-empty compact set A_{β_1,β_2} satisfying

$$A_{\beta_1,\beta_2} = T_{-1}(A_{\beta_1,\beta_2}) \cup T_1(A_{\beta_1,\beta_2}).$$

If $\beta_1 \neq \beta_2$ then the contractions T_i are affine contractions and A_{β_1,β_2} is termed a self-affine set. Since $\beta_1, \beta_2 < 2$, the two contracted copies $T_{-1}(A_{\beta_1,\beta_2})$ and $T_1(A_{\beta_1,\beta_2})$ overlap. There are many fundamental open questions about the structure of overlapping self-affine sets, see for example [5, 6, 7].

The family A_{β_1,β_2} of sets was studied in [7], where Shmerkin proved that there exists an open set $K \subset (1, 2)^2$ such that for almost every pair $(\beta_1, \beta_2) \in K$ the corresponding set A_{β_1,β_2} has positive Lebesgue measure. This was done by studying the absolute continuity of a certain measure defined on A_{β_1,β_2}. In this article we prove that A_{β_1,β_2} contains a neighbourhood of $(0, 0)$ for all $(\beta_1, \beta_2) \in (1, 1 + C)^2$ for some positive constant C which is explicitly defined later.

In fact this problem is closely related to the problem of 'simultaneous β-expansions' studied by Güntürk in [3]. Given $\beta \in (1, 2)$ and $x \in [\frac{-1}{\beta-1}, \frac{1}{\beta-1}]$, a β-expansion of x is a sequence $\underline{a} \in \{-1, 1\}^{\mathbb{N}}$ for which

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = x.$$

This definition can be extended to $\beta > 2$ by letting the digits a_n come from a larger digit set.

For typical x the β-expansion of x is not unique, indeed almost every $x \in [\frac{-1}{\beta-1}, \frac{1}{\beta-1}]$ has uncountably many β-expansions, see [8]. This allows one, given x, to search for β-expansions of x with interesting properties, such as a given digit frequency or that the sequence is a β-expansion of x for more than one β.

In [3], Güntürk proved that given $\beta_1, \beta_2 > 1$ and $(x_1, x_2) \in \mathbb{R}^2$ there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ satisfying

$$\sum_{n=1}^{\infty} a_n \beta_k^{-n} = x_k$$

for each $k \in \{1, 2\}$ whenever a certain algorithm can be implemented, see Proposition 2.1. It was claimed without proof[1] that there exist constants $C, \delta > 0$ such that the algorithm can be implemented whenever $\beta_1, \beta_2 \in (1, 1 + C)$ and $(x_1, x_2) \in (-\delta, \delta)^2$. We prove this fact and provide suitable constants C and δ explicitly. We also prove a number of related results including results on finding β-expansions with given digit frequency and finding sequences which serve as multiple expansions for a range of β_1, β_2. An interesting facet of our work is that the techniques of Güntürk which we use are quite distinct from the usual fractal geometry techniques for studying self-affine sets.

The following is our main theorem.

Theorem 1.1. *There exists a constant $C \approx 0.05$ such that for any $1 < \beta_1 < \beta_2 < 1+C$, there exists $\delta = \delta(\beta_1, \beta_2)$ such that for any pair $(x_1, x_2) \in [-\delta, \delta]^2$, there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$(\sum_{n=1}^{\infty} a_n \beta_1^{-n}, \sum_{n=1}^{\infty} a_n \beta_2^{-n}) = (x_1, x_2). \tag{1}$$

In the self-affine setting, this theorem corresponds to saying that the sequence (a_n) is a coding of the pair (x_1, x_2) in A_{β_1,β_2}, and in particular that $(x_1, x_2) \in A_{\beta_1,\beta_2}$. This leads immediately to the following corollary.

Corollary 1.1. *For all any $1 < \beta_1 < \beta_2 < 1 + C$ we have that the self-affine fractal A_{β_1,β_2} contains a neighbourhood of $(0, 0)$.*

The constant δ is explicitly computable. If β_1 tends to β_2 the constant δ tends to zero.

Remark 1.1. *An important special case of Corollary 1.1 is the case $x_1 = x_2$. This was the main motivation of Güntürk for his original article because of its relevance to analogue digital conversion, see[3]. While in general the constant δ depends on β_1, β_2, in the case that $x_1 = x_2$ we can choose $\delta = 0.16$ independently of β_1, β_2 to give that for all $1 < \beta_1 < \beta_2 < 1 + C$ and $x \in [-0.16, 0.16]$ there exists a sequence $(a_i) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta_1^n} = \sum_{n=1}^{\infty} \frac{a_n}{\beta_2^n}.$$

[1] Güntürk stated in [3] that details would be provided in a later publication, but has confirmed to us that, due to other commitments, no such publication will be forthcoming. Since the techniques of [3] are rather different from the standard techniques for analysing self-affine sets, and the results are interesting, we take the liberty of providing a proof of the stated results of Güntürk in this article.

Using the same techniques, we can also find β-expansions of real numbers which have certain given digit frequencies. It was stated in [3] that the following theorem should follow by suitably adapting the proof of Theorem 1.1, we provide the appropriate adaptation and prove the result giving explicit constants.

Theorem 1.2. *Let $C_1 > 0$ satisfy $(1+C_1)+2(1+C_1)^3 = 6$. Then for all $1 < \beta < 1+C_1$, there exists $\delta = \delta(\beta)$ such that for any $x \in [-\delta, \delta]$ there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ satisfying*

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = x \tag{2}$$

and

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}. \tag{3}$$

One can read off limiting digit frequencies of the sequence (a_n) from equation 3 by noting that

$$\frac{1 - \frac{a_1+a_2+\cdots+a_n}{n}}{2} = \frac{|\{k \in \{1, \cdots n\} : a_k = -1\}|}{n}.$$

Proofs of Theorems 1.1 and 1.2 are given in the next two sections. In the final section we state some further corollaries and remarks.

2 Proof of Theorem 1.1

As stated in the introduction, we are using many of the ideas of [3]. For clarity, we have amalgamated these ideas to form the following proposition, which was proved in [3]. The remainder of our proof of Theorem 1.1, which gives conditions under which the algorithm in Proposition 2.1 can be implemented, is new.

Proposition 2.1. *Given $1 < \beta_1 < \beta_2 < 2$ and $(x_1, x_2) \in \mathbb{R}^2$ suppose that one can implement the following algorithm.*

1. *For $L > 2$ pick real numbers $h_1, \cdots, h_L$ with $h_L \neq 0$ and*
$$h_{L-1} = h_{L-2} = 0,$$
such that β_1, β_2 are roots of the polynomial $P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k}$.
2. *Pick real numbers u_{-L+1}, u_{-L+2} which satisfy the equation*
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h_L \begin{pmatrix} \beta_1^{-1} & \beta_1^{-2} \\ \beta_2^{-1} & \beta_2^{-2} \end{pmatrix} \begin{pmatrix} u_{-L+1} \\ u_{-L+2} \end{pmatrix}$$
Set $u_{-L+3} = \cdots = u_0 = 0$.
3. *Find a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*
$$u_n := \sum_{k=1}^{L} h_k u_{n-k} - a_n$$
satisfies $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Then the sequence $(a_n)_{n=1}^{\infty}$ will satisfy equation (1).

In this article we give rigorous conditions under which the algorithm of Güntürk can be implemented leading to a proof of Theorem 1.1. For completeness we also give the proof of Proposition 2.1. We begin by introducing the polynomial P, P was chosen because it has relatively low degree and satisfies the conditions of Proposition 2.1, but it is likely that better bounds on C and δ can be obtained by choosing a better polynomial P.

Definition 2.1. *Given $\beta_1, \beta_2 > 1$, we define the polynomial P by*

$$P(x) = x^4 - h_1x^3 - h_2x^2 - h_3x - h_4$$

where

$$\begin{aligned} h_1 &= \frac{(\beta_1+\beta_2)(\beta_1^2+\beta_2^2)}{\beta_1^2+\beta_1\beta_2+\beta_2^2} \\ h_2 &= 0 \\ h_3 &= 0 \\ h_4 &= \frac{-(\beta_1\beta_2)^3}{\beta_1^2+\beta_1\beta_2+\beta_2^2}. \end{aligned}$$

We further define the constant C by

$$C := \sqrt[3]{\sqrt{10}-2} \approx 0.05.$$

Lemma 2.1. *The polynomial P satisfies $P(\beta_1) = P(\beta_2) = 0$.*

Proof. Defining,

$$b = \frac{\beta_1\beta_2(\beta_1+\beta_2)}{\beta_1^2+\beta_1\beta_2+\beta_2^2},$$

and

$$c = \frac{(\beta_1\beta_2)^2}{\beta_1^2+\beta_1\beta_2+\beta_2^2}$$

gives us that

$$(x-\beta_1)(x-\beta_2)(x^2+bx+c) = x^4 - h_1x^3 - h_2x^2 - h_3x - h_4 = P(x).$$

Then β_1 and β_2 are roots of P. □

Lemma 2.2. *For $\beta_1, \beta_2 \in (1, 1+C)$ we have that*

$$\sum_{n=1}^{4} |h_k| = |h_1| + |h_4| \leq 2.$$

Proof. Expanding out, we see that

$$\begin{aligned} \sum_{n=1}^{4} |h_k| &= |h_1| + |h_4| \\ &= \frac{(\beta_1+\beta_2)(\beta_1^2+\beta_2^2) + \beta_1^3\beta_2^3}{\beta_1^2+\beta_1\beta_2+\beta_2^2} \\ &\leq \frac{(2+2C)2(1+C)^2 + (1+C)^6}{3} \\ &\leq 2 \end{aligned}$$

whenever $\beta_1, \beta_2 \in (1, 1+C)$, as required. Indeed, C was chosen to be the largest constant such that the above inequalities hold. $\square$

We now prove Theorem 1.1 using Proposition 2.1.

Proof. We set

$$\begin{aligned}
u_{-3} &= \frac{\beta_1^2 + \beta_1\beta_2 + \beta_2^2}{(\beta_2 - \beta_1)\beta_1\beta_2}\left(\frac{x_1}{\beta_2^2} - \frac{x_2}{\beta_1^2}\right), \\
u_{-2} &= \frac{\beta_1^2 + \beta_1\beta_2 + \beta_2^2}{(\beta_2 - \beta_1)\beta_1\beta_2}\left(\frac{x_2}{\beta_1} - \frac{x_1}{\beta_2}\right), \\
u_{-1} &= u_0 = 0.
\end{aligned}$$

These choices of u_i ensure that condition (2) of Proposition 2.1 is satisfied, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h_4 \begin{pmatrix} \beta_1^{-1} & \beta_1^{-2} \\ \beta_2^{-1} & \beta_2^{-2} \end{pmatrix} \begin{pmatrix} u_{-3} \\ u_{-2} \end{pmatrix}.$$

Condition (1) has already been shown to hold for our choice of P by Lemma 2.1. It remains to show that condition (3) holds, i.e. that one can choose some sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that defining u_n for $n \in \mathbb{N}$ by

$$u_n := \sum_{k=1}^{L} h_k u_{n-k} - a_n \tag{4}$$

gives $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$. Since $h_2 = h_3 = 0$ the above equation for u_n becomes

$$u_n = h_1 u_{n-1} + h_4 u_{n-4} - a_n.$$

We set

$$a_n = \begin{cases} -1 & h_1 u_{n-1} + h_4 u_{n-4} < 0 \\ +1 & h_1 u_{n-1} + h_4 u_{n-4} \geq 0 \end{cases}. \tag{5}$$

Now we observe that, if for some $k \in \mathbb{N}$ one has that $u_{k-1}, u_{k-4} \in [-1, 1]$, then it follows from Lemma 2.2 that

$$h_1 u_{k-1} + h_4 u_{k-4} \in [-2, 2].$$

Hence it follows that

$$u_k := h_1 u_{k-1} + h_4 u_{k-4} - a_k \in [-1, 1],$$

and hence by induction that $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Now we define

$$\delta = \frac{\beta_1^2 \beta_2^2 (\beta_2 - \beta_1)}{(\beta_1^2 + \beta_1\beta_2 + \beta_2^2)(\beta_1 + \beta_2)}.$$

We see that $\delta > 0$ whenever $\beta_2 > \beta_1$, but that $\delta \to 0$ as $\beta_2 - \beta_1 \to 0$. From the definition of u_{-3}, u_{-2} we see that for $x_1, x_2 \in [-\delta, \delta]^2$ and $\beta_1, \beta_2 \in (1, 1 + C)$ we have that $u_{-3}, u_{-2} \in [-1, 1]$. Since $u_{-1} = u_0 = 0$ it follows by induction that $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$. Hence conditions $(1), (2)$ and (3) of Proposition 2.1 are satisfied, and so the sequence (a_n) satisfies equation 1 and Theorem 1.1 is proved. $\square$

It remains only to give a formal proof of Proposition 2.1.

Proof. We give a proof for the case $L = 4$, which is the case that we have used. From condition (2), we have that

$$x_i = h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2})$$

for $i = 1, 2$. Rewriting condition (3) gives us that $a_n = \sum_{k=0}^{4} h_k u_{n-k}$. Then summing gives us that

$$\sum_{n=1}^{\infty} a_n \beta_i^{-n} = \sum_{n=1}^{\infty}\sum_{k=0}^{4} h_k u_{n-k}\beta_i^{-n} \tag{6}$$

where $h_0 = -1$. Since the sequence (u_n) is bounded, we have by Fubini's theorem that

$$\sum_{n=1}^{\infty} a_n \beta_i^{-n} = \sum_{k=0}^{4}\sum_{n=1}^{\infty} h_k u_{n-k}\beta_i^{-n} = \sum_{k=0}^{4} h_k \beta_i^{-k} \sum_{n=-k+1}^{\infty} u_n \beta_i^{-n}$$

Here the first equality involved using equation (6) and swapping the order of summation by Fubini. The second equality is just a change of variables. Now, by separating the terms for positive and negative n in the right hand side of the above equation, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \beta_i^{-n} &= \left(\sum_{k=0}^{4} h_k \beta_i^{-k}\right)\left(\sum_{n=1}^{\infty} \frac{u_n}{\beta_i^n}\right) + h_1\beta_i^{-1}u_0 + h_2\beta_i^{-2}(u_0 + u_{-1}\beta_i) \\ &+ h_3\beta_i^{-3}(u_0 + u_{-1}\beta_i + u_{-2}\beta_i^2) + h_4\beta_i^{-4}(u_0 + u_{-1}\beta_i + u_{-2}\beta_i^2 + u_{-3}\beta_i^3).\end{aligned}$$

Since β_i is the root of $P(x)$ we have $\sum_{k=0}^{4} h_k\beta_i^{-k} = 0$ and so the first term vanishes. From conditions (1) and (2), we have $u_{-1} = u_0 = h_2 = h_3 = 0$. Then, removing the zero terms, the right hand side of the above equation becomes $h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2})$, which by condition (2) is equal to x_i. We conclude that

$$\sum_{n=1}^{\infty} a_n \beta_i^{-n} = h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2}) = x_i$$

as required. This completes the proof of Proposition 2.1.

□

Finally we comment that in the case that $x_1 = x_2$ we can give values of δ which are independent of $\beta_1, \beta_2 \in (1, 1+C)$. Our bound on δ was to ensure that $u_{-2}, u_{-3} \in [-1, 1]$. If $x_1 = x_2$ then

$$\begin{aligned}|u_{-2}| \le |u_{-3}| &= \left| x_1 \left(\frac{\beta_1^2 + \beta_1\beta_2 + \beta_2^2}{(\beta_1^3\beta_2^3)}\right)(\beta_1 + \beta_2)\right| \\ &\le |x_1| 6(1+C)^3 \le 1\end{aligned}$$

whenever $|x_1| \le \delta = \frac{1}{6(1+C)^3} \approx 0.16$.

3 β-expansions with a given digit frequency.

With some modifications, the algorithm used in the proof of Proposition 2.1 can also be utilized to prove Theorem 1.2. The following is analagous to Proposition 2.1.

Proposition 3.1. *Given $1 < \beta < 2$ and $x \in [-\delta, \delta]$ for some δ which will be set in the process of proof, suppose that one can implement the following algorithm.*

1. *For $L > 2$ pick real numbers $h_1, \cdots, h_L$ with $h_L \neq 0$ and*

$$h_{L-1} = h_{L-2} = 0,$$

such that $1, \beta$ are roots of the polynomial $P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k}$.

2. *Pick a real number u_{-L+1}, which satisfies the equation*

$$x = \frac{h_L(\beta - 1)u_{-L+1}}{\beta(\beta - 2)}.$$

Set $u_{-L+2} = \cdots = u_0 = 0$.

3. *Find a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$u_n := \left(\sum_{k=1}^{L} h_k u_{n-k} \right) + x - a_n$$

satisfies $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Then the sequence $(a_n)_{n=1}^{\infty}$ satisfies equations (2) and (3).

Such sequences are known as 'hybrid encoders'. We begin by proving Proposition 3.1, this is similar to the proof of Proposition 2.1.

Proof. We begin by rearranging condition (3) of Proposition 3.1 to give

$$a_n = \left(\sum_{k=1}^{4} h_k u_{n-k} \right) + x - u_n = \left(\sum_{k=0}^{4} h_k u_{n-k} \right) + x.$$

Then we have that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = \sum_{n=1}^{\infty} \sum_{k=0}^{4} h_k u_{n-k} \beta^{-n} + x \sum_{n=1}^{\infty} \frac{1}{\beta^n} \tag{7}$$

where $h_0 = -1$. We now follow the reasoning of the proof of Proposition 2.1 exactly, to yield that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = h_4(u_{-3}\beta^{-1} + u_{-2}\beta^{-2}) + \frac{x}{\beta - 1}.$$

Unlike in Proposition 2.1, we also have that $u_{-2} = 0$, so we conclude that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = h_4(u_{-3}\beta^{-1}) + \frac{x}{\beta - 1},$$

and picking $u_{-3} = \frac{\beta(\beta-2)x}{h_4(\beta-1)}$ yields that

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}.$$

It remains to prove part two of the theorem, that

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now we have from the condition (3) of Proposition 3.1 that

$$\left(\sum_{k=1}^{L} h_k u_{n-k}\right) + x - a_n - u_n = \left(\sum_{k=0}^{L} h_k u_{n-k}\right) + x - a_n = 0.$$

Then

$$\left(\frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k}\right) + x - \frac{1}{N}\sum_{n=1}^{N} a_n = 0.$$

We shall prove that $\sum\limits_{n=1}^{N}\sum\limits_{k=0}^{L} h_k u_{n-k}$ is bounded by some constant independent of N, which will give

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = 0$$

and hence that

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now we have that

$$\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = (u_1 + u_2 + \cdots + u_{N-L})(h_0 + h_1 + \cdots + h_L) + \text{ extra terms},$$

where there are $N(L+1) - ((N-L)(L+1)) = L(L+1)$ extra terms, each of which are bounded in absolute value by

$$(\max_{k\in 0,\cdots,L} |h_k|)(\sup_{n\in\mathbb{N}} u_n) \leq \max_{k\in 0,\cdots,L} |h_k| \leq M$$

for some constant M. But $h_0 + h_1 + \cdots + h_L = 0$, and so we see that

$$\left|\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k}\right| \leq 0 + L(L+1)M$$

which is independent of N, and so

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = 0$$

as required. □

Now we prove Theorem 1.2.

Proof. The proof of $\sum\limits_{n=1}^{\infty} a_n\beta^{-n} = x$ is almost the same as the proof of Theorem 1.1. We follow the construction of Lemma 2.1 replacing β_1 by 1 and β_2 by β. This gives

$$P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k} = (z-1)(z-\beta)(z^2+az+b),$$

where $b = \dfrac{\beta(1+\beta)}{1+\beta+\beta^2}$ and $c = \dfrac{\beta^2}{1+\beta+\beta^2}$.

Then we have $h_1 = \frac{(1+\beta)(1+\beta^2)}{1+\beta+\beta^2}$ and $h_4 = \frac{-\beta^3}{1+\beta+\beta^2}$. We choose C_1 such that

$$\begin{aligned}
\sum_{k=1}^{4} |h_k| &= |h_1| + |h_4| \\
&= \frac{(1+\beta)(1+\beta^2)+\beta^3}{1+\beta+\beta^2} \\
&< \frac{(1+C_1)((1+C_1)^2+1)+(1+C_1)^3}{3} \\
&= 2
\end{aligned}$$

where C_1 is the real root of $\frac{(1+x)((1+x)^2+1)+(1+x)^3}{3} = 2$.

Since $\sum\limits_{k=1}^{4} |h_k| = |h_1| + |h_4| < 2$, we can choose $\delta_0 > 0$ such that $\sum\limits_{k=1}^{4} |h_k| = |h_1| + |h_4| \leq 2 - \delta_0 < 2$, thus for any x satisfying $|x| \in [0, \delta_0]$ we have

$$\sum_{n=1}^{4} |h_k| \;=\; |h_1| + |h_4| \leq 2 - \delta_0 \leq 2 - |x| < 2$$

The next step is to prove the boundness of u_n. Choosing $\delta_1 = \frac{h_4(\beta-1)}{\beta(\beta-2)}$ we have that

$$|u_{-3}| = \left|\frac{\beta(\beta-2)x}{h_4(\beta-1)}\right| \leq 1.$$

Finally, if we take $\delta = \min\{\delta_0\,, \delta_1\}$, then this choice can ensure that

$$\sum_{n=1}^{4} |h_k| = |h_1| + |h_4| \leq 2 - x$$

and $|u_{-3}| \leq 1$ hold simultaneously. We also have that $u_{-2} = u_{-1} = u_0 = 0$. We let the sequence (a_n) be chosen as follows:

$$a_n = \begin{cases} -1 & \sum_{k=1}^{L} h_k u_{n-k} + x < 0 \\ +1 & \sum_{k=1}^{L} h_k u_{n-k} + x \geq 0 \end{cases}. \tag{8}$$

Then by induction we have that $u_n \in [-1, 1]$ for all $n \in \mathbb{N}$, and hence the conditions of Proposition 3.1 are fulfilled and Theorem 1.2 is proved. $\square$

4 Further Remarks

We have the following further remarks.

(i) We have proved that if β_1 and β_2 are very close to 1 then A_{β_1,β_2} has an interior, but it is unlikely that our bounds are optimal, see for example the diagrams in [3]. Our proof was based on choosing an expansion $(a_n)_{n=1}^{\infty}$ of pairs (x_1, x_2) using equation (3). Perhaps by using a more sophisticated algorithm one may hope to gain a truer picture of the conditions under which our technique can be made to work.

(ii) The IFS which we study is a little different to that studied by Shmerkin in [7], since we use digit set $\{-1, 1\}$ rather than $(-\frac{1}{\gamma}, -\frac{1}{\lambda})$ and $(\frac{1}{\gamma}, \frac{1}{\lambda})$. However such changes of digit set do not affect whether the attractor of the corresponding IFS has an interior.

(iii) We note that if $\beta_1\beta_2 > 2$ then A_{β_1,β_2} cannot have an interior. Güntürk gave a volume covering argument to prove this. In fact one can say more, the sets A_{β_1,β_2} fall into the setting of 'self-affine sets of Kakeya type' studied in [5], and so by Theorem 3.3 of that paper we have that

$$\dim_B(A_{\beta_1,\beta_2}) = 1 + \frac{\log \frac{2}{\beta_1}}{\log \beta_2} < 2$$

whenever $\beta_1\beta_2 > 2$ and $1 < \beta_1 < \beta_2 < 2$.

(iv) Our approach to generating sequences $\underline{a}$ which satisfy the conditions of Theorem 1.1 is in some sense dynamical, we have an algorithm which chooses a value of (a_n) based on the vector $(u_n, u_{n-1}, u_{n-2}, u_{n-3})$, and then maps this vector to the vector $(u_{n+1}, u_n, u_{n-1}, u_{n-2})$ and repeats the operation. This system is reminiscent of shift radix systems, see [1], except that we have a displacement by a_n.

Our algorithm is far less simple than corresponding algorithms for generating expansions in the one dimensional case, such as the random β-transformation of [2]. It would be nice to have an analogue of the random β-transformation for the higher dimensional case which produces expansions of pairs (x_1, x_2) in a more direct and understandable way.

(v) One can use Remark 1.1 to consider when, for a specific sequence (a_n) and real number x, there exist β_1, β_2 such that

$$x = \sum_{i=1}^{\infty} a_i \beta_1^{-i} = \sum_{i=1}^{\infty} a_i \beta_2^{-i}.$$

Given $\underline{a} \in \{-1, 1\}^{\mathbb{N}}$ we define the function $f_{\underline{a}} : (1, 2] \to \mathbb{R}$ by

$$f_{\underline{a}}(\beta) = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

The function $f_{\underline{a}}$ is continuous and differentiable. We call a sequence $\underline{a} = (a_n)$ a *simultaneous encoder* of x if there exists there exist $1 < \beta_1 < \beta_2 < 2$ such that $x = f_{\underline{a}}(\beta_1) = f_{\underline{a}}(\beta_2)$. By Remark 1.1, for any $1 < \beta_1 < \beta_2 < 1 + C$ and any $x \in [-0.16, 0.16]$ one can find a simultaneous encoder $\underline{a}$ of x satisfying $x = f_{\underline{a}}(\beta_1) = f_{\underline{a}}(\beta_2)$. By the extreme value theorem, the function $f_{\underline{a}}$ has global extrema in $[\beta_1, \beta_2]$. We let $\beta, \beta_0 \in [\beta_1, \beta_2]$ be the values where the global minimum and global maximum take place. Let $y_1 = f_{\underline{a}}(\beta)$ and $y_2 = f_{\underline{a}}(\beta_0)$. Then by the intermediate value theorem, the sequence $\underline{a}$ is a simultaneous encoder for all $z \in (y_1, y_2)$. Thus, if we define the set

$$E_{\underline{a}} := \{x \in \mathbb{R} : \underline{a} \text{ is a simultaneous encoder of } x\}.$$

then the above argument shows that either $E_{\underline{a}}$ is empty, or is a single point or contains an interval.

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